

Random matrix theory of quantum transport in chaotic cavities with non-ideal leads

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We determine the joint probability density function (JPDF) of reflection eigenvalues in three Dyson's ensembles of normal-conducting chaotic cavities coupled to the outside world through both ballistic and tunnel point contacts. Expressing the JPDF in terms of hypergeometric functions of matrix arguments (labeled by the Dyson index β), we further show that reflection eigenvalues form a determinantal ensemble at $\beta = 2$ and a new type of a Pfaffian ensemble at $\beta = 4$. As an application, we derive a simple analytic expression for the concurrence distribution describing production of orbitally entangled electrons in chaotic cavities with tunnel point contacts when time reversal symmetry is preserved.

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Introduction.—Phase-coherent quantum transport¹ in irregular cavities with fully chaotic classical dynamics exhibits a remarkable statistical universality^{2,3}. Fluctuations of various transport observables (such as electrical or thermal conductance, noise power, transferred charge etc.) are described by universal statistical laws which appear to depend on the fundamental symmetries⁴ of a cavity (such as the absence or presence of time-reversal, spin-rotational, particle-hole and chiral/sublattice symmetries) and the transmission properties of the leads attached to it. Other system-specific microscopic details of the scattering region become irrelevant after appropriate ensemble or energy averaging.

Out of three theoretical frameworks – random matrices^{5,6}, semiclassical^{7,8} and field-theoretic approaches⁹ – devised for a non-perturbative description of transport phenomena in chaotic cavities, the random matrix theory is particularly well positioned to explore the universal aspects of quantum transport.

Since at low temperatures and voltages the transport properties can be related to the scattering matrix $\mathbf{S}(\varepsilon_F)$ of the total system comprised by the cavity and the leads,

$$\mathbf{S}(\varepsilon_F) = \mathbf{1}_N - 2i\pi\mathbf{W}^\dagger(\varepsilon_F\mathbf{1}_M - \mathcal{H} + i\pi\mathbf{W}\mathbf{W}^\dagger)^{-1}\mathbf{W}, \quad (1)$$

it becomes a central object of interest. Equation (1), known as the Heidelberg formula¹⁰, suggests that the symmetries and statistics of the scattering matrix derive from the symmetries and statistics of an $M \times M$ random-matrix-Hamiltonian \mathcal{H} used to mimic a chaotic scattering of a single electron inside the cavity as $M \rightarrow \infty$. The coupling of electron states with the Fermi energy ε_F in the cavity to those in the leads is described by an $M \times N$ deterministic matrix \mathbf{W} , where $N = n_L + n_R$ is the total number of propagating modes (channels) in the left (n_L) and right (n_R) leads.

In *normal*-conducting cavities with broken time-reversal symmetry the $N \times N$ scattering matrix $\mathbf{S} = \mathbf{S}(\varepsilon_F)$ is merely unitary ($\beta = 2$); it becomes unitary symmetric $\mathbf{S} = \mathbf{S}^T$ when both time-reversal and spin-

rotational symmetries are preserved ($\beta = 1$), and unitary self-dual quaternion $\mathbf{S} = \sigma_y \mathbf{S}^T \sigma_y$ if the time-reversal symmetry is preserved but the spin-rotational one is broken ($\beta = 4$). Fluctuations of the scattering matrix are described by the Poisson kernel^{11–13}

$$\Pi_{\mathbf{S}_0}^{(\beta)}(\mathbf{S}) \propto [\det(\mathbf{1}_N - \mathbf{S}_0^\dagger \mathbf{S}_0)]^{\theta_N^{(\alpha)}} |\det(\mathbf{1}_N - \mathbf{S}_0^\dagger \mathbf{S})|^{-2\theta_N^{(\alpha)}}. \quad (2)$$

Here $\alpha = 2/\beta$ is a complementary symmetry parameter, and $\theta_N^{(\alpha)} = (N - 1)/\alpha + 1$. The notation “det” stands for a conventional determinant for $\beta = 1$ and 2; it should be interpreted as a quaternion determinant⁵ for $\beta = 4$. Equation (2) highlights the universal character of scattering matrix fluctuations: it depends on the fundamental system symmetries encoded in the Dyson index β and parameters of point contacts contained in the average scattering matrix \mathbf{S}_0 . The $(n_L + n_R)$ eigenvalues $\hat{\gamma} = (\hat{\gamma}_L, \hat{\gamma}_R) = \text{diag}(\{\sqrt{1 - \Gamma_j}\})$ of \mathbf{S}_0 characterise² couplings between the cavity and the leads in terms of tunnel probabilities Γ_j of the j -th electron mode in the leads.

In chaotic cavities probed via *ballistic* point contacts¹⁴ (so-called ‘ideal leads’ characterized by $\Gamma_j = 1$), the average scattering matrix vanishes and the Poisson kernel degenerates to the uniform distribution. The latter implies that fluctuations of scattering matrices are described by three Dyson's circular ensembles⁵ – COE ($\beta = 1$), CUE ($\beta = 2$) or CSE ($\beta = 4$). The uniformity, in turn, induces nontrivial correlations¹⁵ between reflection eigenvalues $\mathbf{R} = (R_1, \dots, R_n)$ which, together with appropriate unitary rotations [see Eq. (15) below], conveniently parameterize the scattering matrix:

$$P_{(0|0)}^{(\beta)}(\mathbf{R}) \propto |\Delta_n^{(\beta)}(\mathbf{R})| \prod_{j=1}^n (1 - R_j)^{\beta/2 - 1 + \beta\nu/2}. \quad (3)$$

This JPDF refers to chaotic cavities probed via two ideal leads. Here, $\Delta_n(\mathbf{R}) = \prod_{j < k} (R_k - R_j)$ is the Vandermonde determinant with $n = \min(n_L, n_R)$, and the pa-

parameter $\nu = |n_L - n_R|$ accounts for asymmetry between the leads.

Considered though the prism of Landauer-type formulae¹⁶, Eq. (3) appears to be of invaluable operational importance. Based on it, detailed non-perturbative predictions^{2,17} have been made for statistics of electrical conductance^{2,15,18,19}, noise power^{2,18-20} and transferred charge^{18,21} and a surprising link between zero-dimensional theories of quantum transport and the theory of integrable lattices²²⁻²⁴ has been unveiled.

Recently, the interest in random matrix theories of quantum transport has been revived. The advent of topological superconductors²⁵ prompted a series of works^{26,27} where novel *circular* ensembles²⁸ of scattering matrices (complementary to Dyson's COE, CUE and CSE) were defined to reveal universal features of transport phenomena in irregular structures described by random-matrix Hamiltonians of the Bogoliubov-de Gennes and Dirac types. Among other results, various extensions of Eq. (3) were derived therein²⁶⁻²⁸ in the context of thermal and

electrical conductances.

While much progress has been achieved in a non-perturbative description of chaotic cavities probed via ballistic point contacts, the world of *non-ideal (tunnel) couplings* is barely understood from the random-matrix-theory perspective. (Involved mathematical structures^{11,29} lurking behind the Poisson kernel^{12,30} are at the root of the poor knowledge.) This is in stark contrast to remarkable technological developments in the field: chaotic structures with adjustable point contacts were fabricated³¹ long ago, and a possibility to efficiently control degree of dephasing³² was recently demonstrated³³.

Effect of tunnel point contacts on JPDF of reflection eigenvalues.—In this paper, we investigate the influence of non-ideal couplings on statistics of reflection eigenvalues in normal-conducting chaotic cavities belonging to either of the three Dyson symmetry classes ($\beta = 1, 2$ or 4). Focussing on the two-lead geometry and assuming that only one (left) lead is attached via tunnel point contacts, we shall show that the corresponding JPDF is given by the expression

$$P_{(\hat{\gamma}_L | \mathbf{0})}^{(\beta)}(\mathbf{R}) \propto [\det(\mathbb{1}_{n_L} - \hat{\gamma}_L^2)]^{\theta_N^{(\alpha)}} |\Delta_n^{(\beta)}(\mathbf{R})| \prod_{j=1}^n (1 - R_j)^{\beta\nu/2 + \beta/2 - 1} \cdot {}_2\mathcal{F}_1^{(\alpha)} \left(\begin{matrix} \theta_N^{(\alpha)}, \theta_N^{(\alpha)} \\ \theta_{n_L}^{(\alpha)} \end{matrix} \middle| \hat{\gamma}_L^2, \mathbf{R}_* \right). \quad (4)$$

Here, ${}_2\mathcal{F}_1^{(\alpha)}(\cdot | \mathbf{X}, \mathbf{Y})$ is a hypergeometric function^{6,34} of two matrix arguments, $\mathbf{X} = \text{diag}(x_1, \dots, x_M)$ and $\mathbf{Y} = \text{diag}(y_1, \dots, y_M)$; it is defined by the series

$${}_2\mathcal{F}_1^{(\alpha)} \left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix} \middle| \mathbf{X}, \mathbf{Y} \right) = \sum_{\{\lambda: \ell(\lambda) \leq M\}} \frac{1}{|\lambda|!} \frac{[a_1]_{\lambda}^{(\alpha)} [a_2]_{\lambda}^{(\alpha)}}{[b_1]_{\lambda}^{(\alpha)}} \frac{C_{\lambda}^{(\alpha)}(\mathbf{X}) C_{\lambda}^{(\alpha)}(\mathbf{Y})}{C_{\lambda}^{(\alpha)}(\mathbb{1}_n)} \quad (5)$$

running over ordered, non-increasing partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ whose length $\ell(\lambda)$ does not exceed M ; $|\lambda|$ is weight of λ ; a generalized Pochhammer symbol $[a]_{\lambda}^{(\alpha)}$ equals

$$[a]_{\lambda}^{(\alpha)} = \prod_{j=1}^{\ell(\lambda)} \left(a - \frac{j-1}{\alpha} \right)_{\lambda_j}, \quad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}. \quad (6)$$

The function $C_{\lambda}^{(\alpha)}(\mathbf{X})$ is the Jack polynomial³⁴ in the C -normalization⁶, such that $\sum_{|\lambda|=k} C_{\lambda}^{(\alpha)}(\mathbf{X}) = (\text{tr} \mathbf{X})^k$. The vector $\hat{\gamma}_L^2 = (1 - \Gamma_1, \dots, 1 - \Gamma_{n_L})$ accommodates a set of tunnel probabilities $(\Gamma_1, \dots, \Gamma_{n_L})$ quantifying a non-ideal coupling of the left lead; the vector $\mathbf{R}_* = \mathbf{R}$ for $n_L \leq n_R$, and $\mathbf{R}_* = (\mathbf{R} \cup \mathbb{1}_{n_L - n_R})$ otherwise. Equation (4) is our *first main result*.

For generic β , Eq. (4) can be written in a more informative form provided that tunnel probabilities are channel independent, $\hat{\gamma}_L^2 = \gamma^2 \mathbb{1}_{n_L}$. In this case, a hypergeometric

function of two matrix arguments reduces to that of *one* matrix argument, ${}_2\mathcal{F}_1^{(\alpha)}(\cdot | \mathbf{X}, \mathbb{1}_M) = {}_2F_1^{(\alpha)}(\cdot | \mathbf{X})$. The latter is related to the Selberg correlation integral³⁵:

$$S_{n,m}^{(\sigma)}(\lambda_1, \lambda_2; \mathbf{x}) = \int_0^1 dt_1 \cdots \int_0^1 dt_n \times |\Delta_n^{(\sigma)}(\mathbf{t})| \prod_{j=1}^n \left(t_j^{\lambda_1} (1 - t_j)^{\lambda_2} \prod_{k=1}^m (t_j - x_k) \right), \quad (7)$$

where $\mathbf{x} = (x_1, \dots, x_m)$. Straightforward calculations yield³⁶:

$$P_{(\hat{\gamma}_L | \mathbf{0})}^{(\beta)}(\mathbf{R}) \propto (1 - \gamma^2)^{n_L \theta_N^{(\alpha)}} \left[\det \left(\frac{\mathbb{1}_{n_L}}{\mathbb{1}_{n_L} - \gamma^2 \mathbf{R}_*} \right) \right]^{\theta_N^{(\alpha)}} \times S_{n_R/\alpha, n_L}^{(4/\beta)} \left(\frac{2}{\beta} - 1, \frac{2}{\beta} - 1; \frac{\mathbb{1}_{n_L}}{\mathbb{1}_{n_L} - \gamma^2 \mathbf{R}_*} \right) P_{(\mathbf{0} | \mathbf{0})}^{(\beta)}(\mathbf{R}). \quad (8)$$

Here, n_R/α is assumed to be an integer; for $\beta = 1$ symmetry class, this restricts n_R to a set of even integers.

Remarkably, Eq. (8) allows us to derive a *pfaffian* representation of the JPDF of reflection eigenvalues at $\beta = 4$, which is complementary to a previously established²⁹ *determinantal* representation of the JPDF at $\beta = 2$. Indeed, at $\beta = 4$, the Selberg correlation integral $S_{2n_R, n_L}^{(1)}$ is essentially an average product of characteristic polynomials in the Jacobi *orthogonal* ensemble. Applying methods

detailed in Ref.³⁷, we derive³⁶:

$$P_{(\gamma \mathbb{1}_{n_L} | \mathbf{o})}^{(4)}(\mathbf{R}) \propto \prod_{j=1}^n (1 - R_j)^{2\nu+1} \times \Delta_n^3(\mathbf{R}) \text{pf} \left[W_\gamma^{(4)}(R_j, R_k) \right], \quad (9)$$

where

$$W_\gamma^{(4)}(R_j, R_k) = (R_j - R_k) \times {}_2F_1^{(1/2)} \left(\begin{matrix} a, a \\ 3 \end{matrix} \middle| \gamma^2 \begin{pmatrix} R_j & 0 \\ 0 & R_k \end{pmatrix} \right) \quad (10)$$

is an antisymmetric two-point scalar kernel expressed in terms of a hypergeometric function of a 2×2 matrix argument; $a = 2n_R + n_L + 1$. Alternatively, the kernel $W_\gamma^{(4)}(R_j, R_k)$ can be written³⁶ in terms of Jacobi polynomials. This result holds true for n_L even and $n_L \leq n_R$; a generic case will be reported elsewhere. Equations (9) and (10) represent our *second main result*. To the best of our knowledge, the Pfaffian ensemble “ $\Delta^3 \text{pf}(\cdot)$ ” has never been reported in the literature. Finding its correlation functions appears to be a nontrivial problem.

Finally, let us comment on the $\beta = 1$ symmetry class, where one could naïvely anticipate appearance of an average product of characteristic polynomials in the Jacobi symplectic ensemble. To realize that this is *not* the case, we turn to Eq. (8) to observe that the JPDF is now given by the Selberg correlation integral $S_{n_R/2, n_L}^{(4)}$ with n_R kept even. Consulting Eq. (7), we readily conclude that it defines an average of a *square root* of the product of characteristic polynomials. Algebraic structures behind these objects are scarcely studied³⁸.

Concurrence distribution at $\beta = 1$.—The JPDF in the form Eq. (4) lays a basis for a nonperturbative analysis of various quantum transport effects³⁶. As an illustration, let us turn to the problem of orbital entanglement production in a chaotic cavity with preserved time-reversal symmetry.

The proposal to use a cavity as an effective orbital entangler for pairs of non-interacting electrons was put forward by Beenakker and collaborators³⁹ a decade ago. Following these authors, we consider a cavity connected to an electron reservoir at the left and right through two pairs of single channel point contacts. Chaotic scattering entangles the outgoing state in the left channels with that in the right channels. The degree of entanglement is quantified by the concurrence $\mathcal{C} = 2\sqrt{\det(\pi\pi^\dagger)/\text{tr}(\pi\pi^\dagger)}$, where π is the 2×2 matrix $\pi = \sigma_y \mathbf{r} \sigma_y \mathbf{t}^\dagger$. The concurrence nullifies in absence of entanglement and reaches unity for maximally entangled states.

Since single-channel point contacts drive the device into extreme quantum limit, fluctuations in entanglement production are expected to be very strong. Indeed, in the case of *ballistic* point contacts, it was found³⁹ that the average concurrence $\langle \mathcal{C} \rangle = \log(4/e) \approx 0.3863$ compares with its standard deviation $\langle \langle \mathcal{C} \rangle \rangle =$

$\sqrt{2 - (\log 4)^2} \approx 0.2796$. The higher-order cumulants required for an adequate description of concurrence fluctuations can be extracted from the probability density $f_0(\mathcal{C}) = 2/(1 + \mathcal{C})^2$ calculated in Ref.⁴⁰.

The effect of *tunnel* point contacts on concurrence fluctuations has been studied numerically in Ref.⁴¹ where it was argued that the orbital entanglement production can be optimised by increasing asymmetry between transparencies of left and right point contacts. Simulations of concurrence distribution for an entangler with a pair of ballistic and a pair of tunnel point contacts indicated that a family of probability densities $\{f_\gamma(\mathcal{C})\}$ is likely to exhibit a point of intersection at $\mathcal{C} = \mathcal{C}_* \approx 1/3$; compared to the case of ideal couplings, the probability density was enhanced for $\mathcal{C} > \mathcal{C}_*$ and diminished for $\mathcal{C} < \mathcal{C}_*$.

The JPDF of reflection eigenvalues Eq. (4) makes it possible⁴² to support this observation analytically. Expressing the concurrence in terms of reflection eigenvalues,

$$\mathcal{C} = 2 \frac{\sqrt{R_1(1-R_1)R_2(1-R_2)}}{R_1 + R_2 - 2R_1R_2}, \quad (11)$$

and deducing from Eq. (8) that

$$P_{(\gamma \mathbb{1}_2 | \mathbf{o})}^{(1)}(R_1, R_2) = \frac{8}{3\pi} (1 - \gamma^2)^5 |R_1 - R_2| \times \frac{1 + \frac{2}{3}\gamma^2(R_1 + R_2) + \gamma^4 R_1 R_2}{\prod_{j=1}^2 R_j^{1/2} (1 - R_j)^{1/2} (1 - \gamma^2 R_j)^{7/2}}, \quad (12)$$

we derive after some algebra:

$$f_\gamma(\mathcal{C}) = f_0(\mathcal{C}) + \frac{2\gamma^4}{(1 + \mathcal{C})^3} \left(\mathcal{C} - \frac{1}{3} \right). \quad (13)$$

This result locates the intersection point at $\mathcal{C}_* = 1/3$ corroborating numerics of Ref.⁴¹ extremely well.

Sketch of the derivation (JPDF).—The probability density of $n_L \times n_L$ reflection matrix \mathbf{r} is the starting point of our analysis. Substituting the average scattering matrix of the form $\mathbf{S}_0 = (\hat{\gamma}_L \cup 0 \times \mathbb{1}_{n_R})$ into Eq. (2) and eliminating ‘transmission’ degrees of freedom, one obtains:

$$\tilde{P}_{(\hat{\gamma}_L | \mathbf{o})}^{(\beta)}(\mathbf{r}) \propto [\det(\mathbb{1}_{n_L} - \hat{\gamma}_L^2)]^{\theta_N^{(\alpha)}} |\det(\mathbb{1}_{n_L} - \hat{\gamma}_L \mathbf{r})|^{-2\theta_N^{(\alpha)}} \times [\det(\mathbb{1}_{n_L} - \mathbf{r} \mathbf{r}^\dagger)]^{\beta/2 - 1 + \beta\nu/2} \theta(\mathbb{1}_{n_L} - \mathbf{r} \mathbf{r}^\dagger). \quad (14)$$

Reflection eigenvalues are brought into play through a singular value decomposition $\mathbf{r} = \mathbf{u} \hat{\mathbf{g}} \mathbf{v}$, where $\hat{\mathbf{g}} = \mathbf{R}_*^{1/2}$, whilst \mathbf{u} and \mathbf{v} are either constrained ($\beta = 1, 4$) or unconstrained ($\beta = 2$) unitary matrices as specified by Eq. (15) below. To determine the JPDF of reflection eigenvalues, one has to integrate out rotational degrees of freedom with respect to the weighted product of two Haar measures on $U(n_L)$,

$$d\mu^{(\beta)}(\mathbf{u}, \mathbf{v}) = d\mu(\mathbf{u}) d\mu(\mathbf{v}) \begin{cases} \delta(\mathbf{v} - \mathbf{u}^\dagger), & \beta = 1; \\ 1, & \beta = 2; \\ \delta(\mathbf{v} - \sigma_y \mathbf{u}^\dagger \sigma_y), & \beta = 4. \end{cases} \quad (15)$$

This yields:

$$P_{(\hat{\gamma}_L | \mathbf{0})}^{(\beta)}(\mathbf{R}) = [\det(\mathbb{1}_{n_L} - \hat{\gamma}_L^2)]^{\theta_N^{(\alpha)}} P_{(\mathbf{0}, \mathbf{0})}^{(\beta)}(\mathbf{R}) \times \int \frac{d\mu^{(\beta)}(\mathbf{u}, \mathbf{v})}{|\det(\mathbb{1}_{n_L} - \hat{\gamma}_L \mathbf{u} \hat{\gamma}_L \mathbf{v})|^{2\theta_N^{(\alpha)}}}. \quad (16)$$

The group integral⁴³,

$$I_{(p,q)}^{(\beta)}(\mathbf{A}, \mathbf{B}) = \int \frac{d\mu^{(\beta)}(\mathbf{u}, \mathbf{v})}{|\det(\mathbb{1}_p - \mathbf{A}\mathbf{u}\mathbf{B}\mathbf{v})|^{2q}}, \quad (17)$$

appearing in Eq. (16), has previously been calculated^{29,44} for $\beta = 2$ only. To evaluate it for other β 's, we expand an inverse determinant

$$\det^{-q}(\mathbb{1}_p - \mathbf{A}\mathbf{u}\mathbf{B}\mathbf{v}) = \sum_{\{\lambda, \ell(\lambda) \leq p\}} \frac{[q]_{\lambda}^{(\alpha)}}{|\lambda|!} C_{\lambda}^{(\alpha)}(\mathbf{A}\mathbf{u}\mathbf{B}\mathbf{v}) \quad (18)$$

in terms of Jack polynomials $C_{\lambda}^{(\alpha)}$ to arrive at the series

$$I_{(p,q)}^{(\beta)}(\mathbf{A}, \mathbf{B}) = \sum_{\substack{\{\lambda, \ell(\lambda) \leq p\} \\ \{\mu, \ell(\mu) \leq p\}}} \frac{[q]_{\lambda}^{(\alpha)} [q]_{\mu}^{(\alpha)}}{|\lambda|! |\mu|!} \tilde{I}_{\lambda, \mu}^{(\beta)}(\mathbf{A}, \mathbf{A}; \mathbf{B}), \quad (19)$$

where

$$\tilde{I}_{\lambda, \mu}^{(\beta)}(\mathbf{A}_1, \mathbf{A}_2; \mathbf{B}) = \int d\mu^{(\beta)}(\mathbf{u}, \mathbf{v}) C_{\lambda}^{(\alpha)}(\mathbf{A}_1 \mathbf{u} \mathbf{B} \mathbf{v}) \overline{C_{\mu}^{(\alpha)}(\mathbf{A}_2 \mathbf{u} \mathbf{B} \mathbf{v})}. \quad (20)$$

The latter integral can be calculated using insights from Ref.⁴⁵. Deferring details of the proof to a separate publication³⁶, we state the result:

$$\tilde{I}_{\lambda, \mu}^{(\beta)} = \delta_{\lambda \mu} \frac{|\lambda|!}{[\theta_p^{(\alpha)}]_{\lambda}^{(\alpha)}} \frac{C_{\lambda}^{(\alpha)}(\mathbf{B}\mathbf{B}^{\dagger})}{C_{\lambda}^{(\alpha)}(\mathbb{1}_p)} C_{\lambda}^{(\alpha)}(\mathbf{A}_1 \mathbf{A}_2^{\dagger}). \quad (21)$$

Combining Eqs. (5), (17) – (21) one concludes that

$$I_{(p,q)}^{(\beta)}(\mathbf{A}, \mathbf{B}) = {}_2\mathcal{F}_1^{(\alpha)} \left(\begin{matrix} q, q \\ \theta_p^{(\alpha)} \end{matrix} \middle| \mathbf{A}\mathbf{A}^{\dagger}, \mathbf{B}\mathbf{B}^{\dagger} \right). \quad (22)$$

Substituting it back to Eq. (16) completes our derivation of the JPDF of reflection eigenvalues [Eq. (4)].

Summary.—In this paper, we formulated a random-matrix-theory approach to quantum transport in normal-conducting chaotic structures probed through both ballistic and tunnel point contacts. Starting with the Poisson kernel, we calculated the JPDF of reflection eigenvalues for arbitrary Dyson's index β and showed that it is expressed in terms of a hypergeometric function of matrix arguments. This general result implies that reflection eigenvalues form a “ $\Delta \det(\cdot)$ ” determinantal process for $\beta = 2$ and a novel “ $\Delta^3 \text{pf}(\cdot)$ ” pfaffian process for $\beta = 4$. Although finding an algebraically insightful representation for the JPDF at $\beta = 1$ remains an open problem, a hypergeometric representation of JPDF still is of operational value: as an example, we calculated the concurrence distribution for orbital entanglement production in chaotic cavities with asymmetric left and right point contacts.

Finally, let us point out that a general framework outlined in the paper lays a basis for many more – both mathematics- and physics-oriented – studies. In particular, analysis⁴⁶ of quantum transport effects in superconducting chaotic cavities^{26,30} and an exact solution³⁶ of Büttiker's dephasing model³² will be reported elsewhere.

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- ¹ Y. Imry, *Introduction to Mesoscopic Physics* (Oxford University Press, New York, 2002).
² C. W. J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).
³ C. Alhassid, Rev. Mod. Phys. **72**, 895 (2000).
⁴ M. R. Zirnbauer, J. Math. Phys. **37**, 4986 (1996); A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).
⁵ M. L. Mehta, *Random Matrices* (Amsterdam: Elsevier, 2004).
⁶ P. J. Forrester, *Log Gases and Random Matrices* (Princeton: Princeton University Press, 2010).
⁷ K. Richter, *Semiclassical Theory of Mesoscopic Quantum Systems* (Berlin: Springer, 2000).
⁸ D. Waltner, *Semiclassical Approach to Mesoscopic Systems* (Berlin: Springer, 2012).
⁹ K. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, 1997).

- ¹⁰ C. Mahaux and H. A. Weidenmüller, *Shell-Model Approach to Nuclear Reactions* (North-Holland, Amsterdam, 1969).
¹¹ L. K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains* (American Mathematical Society, Providence, 1963).
¹² P. W. Brouwer, Phys. Rev. B **51**, 16878 (1995).
¹³ P. A. Mello and H. U. Baranger, Waves Random Media **9**, 105 (1999).
¹⁴ H. van Houten and C. Beenakker, Physics Today **49**, 22 (1996).
¹⁵ H. U. Baranger and P. A. Mello, Phys. Rev. Lett. **73**, 142 (1994); R. A. Jalabert, J.-L. Pichard, and C. W. J. Beenakker, Europhys. Lett. **27**, 255 (1994).
¹⁶ R. Landauer, J. Res. Dev. **1**, 223 (1957); D. S. Fisher and P. A. Lee, Phys. Rev. B **23**, R6851 (1981); M. Büttiker, Phys. Rev. Lett. **65**, 2901 (1990).

- ¹⁷ C. W. J. Beenakker, in: *The Oxford Handbook of Random Matrix Theory*, edited by G. Akemann, J. Baik, and P. Di Francesco (Oxford University Press, 2011).
- ¹⁸ D. V. Savin, H.-J. Sommers and W. Wieczorek, Phys. Rev. B **77**, 125332 (2008).
- ¹⁹ B. A. Khoruzhenko, D. V. Savin and H.-J. Sommers, Phys. Rev. B **80**, 125301 (2009).
- ²⁰ D. V. Savin and H.-J. Sommers, Phys. Rev. B **73**, R081307 (2006).
- ²¹ M. Novaes, Phys. Rev. B **78**, 035337 (2008).
- ²² V. Al. Osipov and E. Kanzieper, Phys. Rev. Lett. **101**, 176804 (2008); J. Phys. A: Math. Theor. **42**, 475101 (2009).
- ²³ F. Mezzadri and N. J. Simm, Commun. Math. Phys. **324**, 465 (2013).
- ²⁴ E. Kanzieper, Constructive Approximation **41**, 615 (2015).
- ²⁵ A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008); A. Kitaev, in: *Advances in Theoretical Physics*, AIP Conf. Proc. **1134**, 22 (2009).
- ²⁶ J. P. Dahlhaus, B. Béri, and C. W. J. Beenakker, Phys. Rev. B **82**, 014536 (2010); C. W. J. Beenakker, J. P. Dahlhaus, M. Wimmer, and A. R. Akhmerov, Phys. Rev. B **83**, 085413 (2011); M. Marciani, P. W. Brouwer, and C. W. J. Beenakker, Phys. Rev. B **90**, 045403 (2014).
- ²⁷ A. F. Macedo-Jun. and A. M. S. Macêdo, Phys. Rev. B **66**, R041307 (2002); M. S. M. Barros, A. J. Nascimento-Jun., A. F. Macedo-Jun., J. G. G. S. Ramos, and A. L. R. Barbosa, Phys. Rev. B **88**, 245133 (2013).
- ²⁸ C. W. J. Beenakker, arXiv: 1407.2131 [Rev. Mod. Phys. (to be published)].
- ²⁹ P. Vidal and E. Kanzieper, Phys. Rev. Lett. **108**, 206806 (2012).
- ³⁰ B. Béri, Phys. Rev. B **79**, 214506 (2009).
- ³¹ See, e.g., L. P. Kouwenhoven, C. M. Marcus, P. L. McEuen, S. Tarucha, R. M. Westervelt, and N. S. Wingreen, in: *Mesoscopic Electron Transport*, edited by L. L. Sohn, L. P. Kouwenhoven, and G. Schön (Kluwer, Dordrecht, 1997).
- ³² Handling of non-ideal couplings is essential to simulate phase-breaking effects in the framework of voltage-probe model. See, e.g., M. Büttiker, Phys. Rev. B **33**, 3020 (1986); H. U. Baranger and P. A. Mello, Phys. Rev. B **51**, R4703 (1995); P. W. Brouwer and C. W. J. Beenakker, Phys. Rev. B **55**, 4695 (1997).
- ³³ P. Roulleau, F. Portier, P. Roche, A. Cavanna, G. Faini, U. Gennser, and D. Mailly, Phys. Rev. Lett. **102**, 236802 (2009).
- ³⁴ I. G. Macdonald, *Symmetric Functions and Hall Polynomials* (Clarendon Press, Oxford, 1995).
- ³⁵ J. Kaneko, SIAM J. Math. Anal. **24**, 1086 (1993).
- ³⁶ A. Jarosz, P. Vidal, and E. Kanzieper, unpublished (2014).
- ³⁷ A. Borodin and E. Strahov, Commun. Pure and Appl. Math. **59**, 161 (2006).
- ³⁸ Y. V. Fyodorov and A. Nock, J. Stat. Phys. **159**, 731 (2015); G. Akemann, T. Guhr, M. Kieburg, R. Wegner, and T. Wirtz, Phys. Rev. Lett. **113**, 250201 (2014); R. D. Gupta and D. St. P. Richards, SIAM J. Math. Anal. **16**, 852 (1985).
- ³⁹ C. W. J. Beenakker, M. Kindermann, C. M. Marcus, and A. Jacoby, in: *Fundamental Problems of Mesoscopic Physics*, edited by I. V. Lerner, B. L. Altshuler, and Y. Gefen (Kluwer Academic Publishers, 2004).
- ⁴⁰ V. A. Gopar and D. Frustaglia, Phys. Rev. B **77**, 153403 (2008).
- ⁴¹ F. A. G. Almeida and A. M. C. Souza, Phys. Rev. B **82**, 115422 (2010).
- ⁴² D. Villamaina and P. Vivo, Phys. Rev. B **88**, R041301 (2013).
- ⁴³ In Eq. (17) and below, matrices **A** and **B** are assumed to be arbitrary complex ($\beta = 2$), complex symmetric ($\beta = 1$), or self-dual quaternion ($\beta = 4$).
- ⁴⁴ A. Yu. Orlov, Int. J. Mod. Phys. A **19**, 276 (2004).
- ⁴⁵ Z. M. Feng and J. P. Song, J. Phys. A: Math. Theor. **42**, 325204 (2009).
- ⁴⁶ P. Vidal, arXiv: 1411.4746 [J. Phys. A (to be published)].